Searching for proofs (and uncovering capacities of the mathematical mind)[1]

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Abstract. What is it that shapes mathematical arguments into proofs that are intelligible to us, and what is it that allows us to find proofs efficiently? — This is the informal question I intend to address by investigating, on the one hand, the abstract ways of the axiomatic method in modern mathematics and, on the other hand, the concrete ways of proof construction suggested by modern proof theory. These theoretical investigations are complemented by experimentation with the proof search algorithm AProS. It searches for natural deduction proofs in pure logic; it can be extended directly to cover elementary parts of set theory and to find abstract proofs of Gödel's incompleteness theorems. The subtle interaction between understanding and reasoning, i.e., between *introducing concepts* and *proving theorems*, is crucial. It suggests principles for structuring proofs conceptually and brings out the dynamic role of *leading ideas*. Hilbert's work provides a perspective that allows us to weave these strands into a fascinating intellectual fabric and to connect, in novel and surprising ways, classical themes with deep contemporary problems. The connections reach from proof theory through computer science and cognitive psychology to the philosophy of mathematics and all the way back.

1 Historical perspective

It is definitely counter to the standard view of Hilbert's formalist perspective on mathematics that I associate his work with uncovering aspects of the mathematical mind; I hope you will see that he played indeed a pivotal role. He was deeply influenced by Dedekind and Kronecker; he connected these extraordinary mathematicians of the 19th century to two equally remarkable logicians of the 20th century, Gödel and Turing. The character of that connection is determined by Hilbert's focus on the *axiomatic method* and the associated *consistency problem*. What a remarkable path it is: emerging from the radical transformation of mathematics in the second half of the 19th century and leading to the dramatic development of metamathematics in the second half of the 20th century.

Examining that path allows us to appreciate Hilbert's perspective on the wide-open mathematical landscape. It also enriches our perspective on his

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[1] This essay is dedicated to Grigori Mints on the occasion of his 70th birthday. Over the course of many years we have been discussing the fruitfulness of searching directly for natural deduction proofs. He and his Russian colleagues took already in 1965 a systematic and important step for propositional logic; see the co-authored paper (Shanin, et al. 1965), but also (Mints 1969) and the description of further work in (Maslov, Mints, and Orevkov 1983).

metamathematical work.[1] Some of Hilbert's considerations are, however, not well integrated into contemporary investigations. In particular, the *cognitive* side of proof theory has been neglected, and I intend to pursue it in this essay. It was most strongly, but perhaps somewhat misleadingly, expressed in Hilbert's Hamburg talk of 1927. He starts with a general remark about the "formula game" criticized by Brouwer:

The formula game ... has, besides its mathematical value, an important general philosophical significance. For this formula game is carried out according to certain definite rules, in which the technique of our thinking is expressed. These rules form a closed system that can be discovered and definitively stated.

Then he continues with a provocative statement about the cognitive goal of proof theoretic investigations.

The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds.[2]

It is clear to us, and it was clear to Hilbert, that mathematical thinking does not proceed in the strictly regimented ways imposed by an austere formal theory. Though formal rigor is crucial, it is not sufficient to shape proofs intelligibly or to discover them efficiently, even in pure logic. Recalling the principle that mathematics should solve problems "by a minimum of blind calculation and a maximum of guiding thought", I will investigate the subtle interaction between understanding and reasoning, i.e., between *introducing concepts* and *proving theorems*. That suggests principles for structuring proofs conceptually and brings out the dynamic role of *leading ideas*.[3]

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[1] In spite of the demise of the finitist program, proof theoretic work has been continued successfully along at least two dimensions. There is, first of all, the ever more refined formalization of mathematics with the novel mathematical end of extracting information from proofs. Formalizing mathematics was originally viewed as the basis for a mathematical treatment of foundational problems and, in particular, for obtaining consistency results. Gödel's theorems shifted the focus from absolute finitist to relative consistency proofs with the philosophical end of comparing foundational frameworks; that is the second dimension of continuing proof theoretic work. These two dimensions are represented by "proof mining" initiated by Kreisel and "reductive proof theory" pursued since Gödel and Gentzen's consistency proof of classical relative to intuitionistic number theory.

[2] (Hilbert 1927) in (van Heijenoort 1967, p. 475).

[3] The way in which I am pursuing matters is programmatically related to Wang's perspective in his (1970). In that paper Wang discusses, on p. 106, "the project of mechanizing mathematical arguments". The results that have been obtained so far, Wang asserts, are only "theoretical" ones, "which do not establish the strong conclusion that mathematical reasoning (or even a major part of it) is mechanical in nature". But the unestablished strong conclusion challenges us to address in novel ways "the perennial problem about mind and machine" — by dealing with mathematical activity in a systematic way. Wang continues: "Even though what is demanded is not mechanical simulation, the task requires a close examination of how mathematics is done in order to determine how informal methods can be replaced by mechanizable procedures and how the speed of computers can be employed to compensate for their inflexibility. The field is wide open, and like all good things, it is not easy. But one does expect and look for pleasant surprises in this enterprise which requires a novel combination of psychology, logic, mathematics and computer technology." Surprisingly, there is still no unified interdisciplinary approach; but see Appendix C below with the title "Confluence?".

In some sense, the development toward proof theory began in late 1917 when Hilbert gave a talk in Zürich, entitled *Axiomatisches Denken*. The talk was deeply rooted in the past and pointed decisively to the future. Hilbert suggested, in particular,

... we must — that is my conviction — take the concept of the specifically mathematical proof as an object of investigation, just as the astronomer has to consider the movement of his position, the physicist must study the theory of his apparatus, and the philosopher criticizes reason itself.

Hilbert recognized, in the very next sentence, that "the execution of this program is at present, to be sure, still an unsolved problem". Ironically, solving this problem was just a step in solving the most pressing issue with modern abstract mathematics as it had emerged in the second half of the 19th century. This development of mathematics is complemented by and connected to the dramatic expansion of logic facilitating steps toward full formalization.[4] Hilbert clearly hoped to address the issue he had already articulated in his Paris address of 1900 and had stated prominently as the second in his famous list of problems:

... I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms [of arithmetic]: *To prove that they are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results.*

As to the axioms of arithmetic, Hilbert points to his paper *Über den Zahlbegriff* delivered at the Munich meeting of the German Association of Mathematicians in September of 1899. The title alone indicates already its intellectual context: twelve years earlier, Kronecker had published a well-known paper with the very same title and had sketched a way of introducing irrational numbers without accepting the general notion. It is precisely to the general concept that Hilbert wants to give a proper foundation — using the axiomatic method and following Dedekind who represents most strikingly the development toward greater abstractness in mathematics.

2 Abstract concepts

Howard Stein analyzed philosophical aspects of the 19th century expansion and transformation of mathematics I just alluded to.[5] Underlying these develop-

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[4]The deepest philosophical connection between the mathematical and logical developments is indicated by the fact that both Dedekind and Frege considered the concept of a "function" to be central; it is a dramatic break from traditional metaphysics. Cf. Cassirer's *Substanzbegriff und Funktionsbegriff*.

[5]Stein did so in his marvelous paper (Stein 1988). The key words of its title (logos, logic, and logistiké) structure the systematic progression of my essay that was presented as the Howard Stein Lecture at the University of Chicago on 15 May 2008; Part 2 is a discussion of logos, Part 3 of logic, and Part 4 of logistiké. Improved versions of that talk were presented on 8 October 2008 to a workshop on "Mathematics between the Natural Sciences and the Humanities" held in Göttingen, on 28 December 2008 to the Symposium on "Hilbert's Place in the Foundations and Philosophy of Mathematics" at the meeting of the American Philosophical Association in Philadelphia, on 27 February 2009 in the series "Formal Methods in the Humanities" at Stanford University, and on 16 April 2009 to the conference on "The Fundamental Idea of Proof Theory" in Paris. I am grateful to many

ments is for him the rediscovery of a *capacity of the human mind* that had been first discovered by the Greeks between the 6th and 4th century B.C.:

The expansion [of mathematics in the 19th century] was effected by the very same capacity of thought that the Greeks discovered; but in the process, something new was learned about the nature of that capacity — what it is, and what it is not. I believe that what has been learned, when properly understood, constitutes one of the greatest advances in philosophy — although it, too, like the advance in mathematics itself, has a close relation to ancient ideas.[6]

The deep connections and striking differences between the two discoveries can be examined by comparing Eudoxos's theory of proportion with Dedekind's and Hilbert's theory of real numbers. Fundamental for articulating this difference is Dedekind's notion of *system* that is also used by Hilbert.

2.1   Systems

When discussing Kronecker's demand that proofs be constructive and that notions be decidable, Stein writes:

I think the issue concerns definitions rather more crucially than proofs; but let me say, borrowing a usage from Plato, that it concerns the mathematical *logos*, in the sense both of 'discourse' generally, and of definition — i.e., the formation of concepts — in particular. (p. 251)

Logos refers to definitions not only as abbreviatory devices, but also as providing a frame for discourse, here the discourse concerning irrational numbers. Indeed, the frame is provided by a *structural* definition that concerns systems and that imposes relations between their elements. This methodological perspective shapes Dedekind's mathematical and foundational work, and Hilbert clearly stands in this Dedekindian tradition. The structural definitions of Euclidean space in Hilbert's (1899a) and of real numbers in his (1900b) start out with, *We think three systems of things...* , respectively with *We think a system of things; we call these things numbers and denote them by a, b, c ... We think these numbers in certain mutual relations, the precise and complete description of which is given by the following axioms: ...* [7] The last sentence is followed by the conditions characterizing real numbers, i.e., those of Dedekind's (1872c), except that continuity is postulated in a different, though deeply related way (see below). Hilbert and Bernays called this way of giving a structural definition, or formulating a mathematical theory, *existential axiomatics*.

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remarks from the various audiences. The final version of this essay was influenced by very helpful comments from two anonymous referees and Sol Feferman. — Dawn M[c]Laughlin prepared the L^AT_EX version of this document; many thanks to her for her meticulous attention to detail.

[6](Stein 1988, pp. 238–239). Stein continues: "I also believe that, when properly understood, this philosophical advance should conduce to a certain modesty: one of the things we should have learned in the course of it is how much we do *not* yet understand about the nature of mathematics." — I could not agree more.

[7]The German texts are: "*Wir denken drei Systeme von Dingen ...*, respectively *Wir denken ein System von Dingen; wir nennen diese Dinge Zahlen und bezeichnen sie mit a, b, c ... Wir denken diese Zahlen in gewissen gegenseitigen Beziehungen, deren genaue und vollständige Beschreibung durch die folgenden Axiome geschieht: ...*"

The introduction of concepts "rendered necessary by the frequent recurrence of complex phenomena, which could be controlled only with difficulty by the old ones" is praised by Dedekind as the engine of progress in mathematics and other sciences.[8] The definition of *continuity* or *completeness* in his (1872c) is to be viewed in this light. The underlying complex phenomena are related to orderings. Dedekind emphasizes transitivity and density as central properties of an ordered system O, and adds the feature that every element in O generates a cut; a *cut* of O is simply a partition of O into two non-empty parts A and B, such that all the elements of A are smaller than all the elements of B. Two different interpretations are presented for these principles, namely, the rational numbers with the ordinary less-than relation and the geometric line with the to-the-right-of relation. On account of this fact the ordering phenomena for the rationals and the geometric line are viewed as *analogous*. Finally, the *continuity principle* is the converse of the last condition: every cut of the ordered system is produced by exactly one element. For Dedekind this principle expresses the essence of continuity and holds for the geometric line.[9]

In order to capture continuity arithmetically and to define a system of real numbers, Dedekind turns the analogy between the rationals and the geometric line into a *real correspondence* by embedding the rationals into the line (after having fixed an origin and a unit). This makes clear that the system of rationals is not continuous, and it motivates considering cuts of rationals as arithmetic counterparts to geometric points. Dedekind shows the system of these cuts to be an ordered field that is also continuous or complete.[10] The completeness of the system, its non-extendibility, points to the core of the difference with Eudoxos's definition of proportionality in Book V of Euclid's *Elements*. The ancient definition applies to many different kinds of geometric magnitudes without requiring that their respective systems be complete, as they may be open to new geometric constructions. Hilbert's completeness axiom expresses the condition of non-extendibility most directly as part of the structural definition. As a matter of fact, even in his (1922) Hilbert articulates Dedekind's structural way of thinking of the system of real numbers when describing the *axiomatische Begründungsmethode* for analysis (that is done still before finitist proof theory is given its proper methodological articulation in 1922):

The continuum of real numbers is a system of things, which are linked to one another by determinate relations, the so-called axioms. In particular, in place of the definition of real numbers by Dedekind cuts, we have the two axioms of continuity, namely, the

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[8] (Dedekind 1888, p. VI).

[9] Dedekind remarks on p. 11 of (1872c): "Die Annahme dieser Eigenschaft der Linie ist nichts als ein Axiom, durch welches wir erst der Linie ihre Stetigkeit zuerkennen, durch welches wir die Stetigkeit in die Linie hineindenken." Then he continues that the "really existent" space may or may not be continuous and that — even if it were not continuous — we could make it continuous in thought. On p. VII of (1888) he discusses a model for Euclid's *Elements* that is everywhere discontinuous.

[10] My interpretation of these considerations reflects Dedekind's methodological practice that is tangible in (1872c) and perfectly explicit five years later in his (1877) — with reference back to (1872c). Thus, Noether attributed the "axiomatische Auffassung" to Dedekind in her comments on (1872c). Notice that Dedekind does *not* identify real numbers with cuts of rationals; real numbers are associated with or determined by cuts, but are viewed as *new* objects. That is vigorously expressed in letters to Lipschitz.

Archimedean axiom and the so-called completeness axiom. To be sure, Dedekind cuts can then also be used to specify individual real numbers, but they do not provide the definition of the concept of real number. Rather, a real number is conceptually just a thing belonging to our system. . . .

This standpoint is logically completely unobjectionable, and the only thing that remains to be decided is, whether a system of the requisite sort is thinkable, that is, whether the axioms do not, say, lead to a contradiction.[11]

The axioms serve, of course, also as starting-points for the systematic development of analysis; consistency is to ensure that not too much can be proved, namely, everything. This is one of the crucially important connections to provability. Dedekind also points repeatedly and polemically to the fact that we have finally a proof of $\sqrt{3}\sqrt{2} = \sqrt{6}$ and indicates how analysis can be developed; he shows the continuity principle to be *equivalent* to the basic analytic fact that bounded, monotonically increasing functions have a limit. That is methodology par excellence: The continuity principle is not only sufficient to prove the analytic fact, but indeed necessary.

2.2   Consistency

For both Dedekind and Hilbert, the coherence of their theories for real numbers was central. Dedekind had aimed for, and thought he had achieved in his (1888), "the purely logical construction of the science of numbers and the continuous realm of numbers gained in it."[12] Within the logical frame of that essay Dedekind defines simply infinite systems and provides also an "example" or "instance". The point of such an instantiation is articulated sharply and forcefully in his famous letter to Keferstein where he asks, whether simply infinite systems "exist at all in the realm of our thoughts". He supports the affirmative answer by a logical existence proof. Without such a proof, he explains, "it would remain doubtful, whether the concept of such a system does not perhaps contain internal contradictions". His *Gedankenwelt*, "the totality S of all things that can be object of my thinking", was crucial for obtaining a simply infinite system.[13]

Cantor recognized Dedekind's *Gedankenwelt* as an inconsistent system and communicated that fact to both Dedekind (in 1896) and to Hilbert (in 1897). When Hilbert formulated arithmetic in his (1900b), he reformulated the

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[11](Hilbert 1922) in (Ewald 1996, p. 1118). — That is fully in Dedekind's spirit: Hilbert's critical remark about the definition of real numbers as cuts do not apply to Dedekind, as should be clear from my discussion (in the previous note), and the issue of consistency was an explicit part of Dedekind's logicist program.

[12]The systematic build-up of the continuum envisioned in (1872c, pp. 5–6) is carried out in later manuscripts where integers and rationals are introduced as equivalence classes of pairs of natural numbers; they serve as models for subsystems of the axioms for the reals, in a completely modern way. — All of these developments as well as that towards the formulation of simply infinite systems are analyzed in (Sieg 2005).

[13]Let me support, by appeal to authority, the claim that Dedekind's thoughts are not psychological ideas: Frege asserts in his manuscript *Logik* from 1894 that he uses the word "Gedanke" in an unusual way and remarks that "Dedekind's usage agrees with mine". It is worthwhile noting that Frege, in this manuscript, approved of Dedekind's argument for the existence of an infinite system. — Note also that Hilbert formulated his existential axiomatics with the phrase "wir denken", so that the system is undoubtedly an object of our thought, indeed, "ein Gedanke".

problem of instantiating logoi as a quasi-syntactic problem: Show that no con-
tradiction is provable from the axiomatic conditions in a finite number of logical
steps. That is, of course, the second problem of his Paris address I discussed
in section 1. He took for granted that consistency amounts to mathematical
existence and assumed that the ordinary investigations of irrational numbers
could be turned into a model theoretic consistency proof within a restricted
logicist framework. This was crucial for the arithmetization of analysis and
its logicist founding. It should be mentioned that Hilbert in *Grundlagen der
Geometrie* also "geometrized" analysis by giving a geometric model via his
"Streckenrechnung" for the axioms of arithmetic (with full continuity only in
the second edition of the *Grundlagen* volume).

In his lecture (*1920b), Hilbert formulated the principles of Zermelo's set
theory (in the language of first-order logic). He considered Zermelo's theory
as providing the mathematical objects Dedekind had obtained through logicist
principles; Hilbert remarked revealingly:

The theory, which results from developing all the consequences of this axiom system,
encompasses all mathematical theories (like number theory, analysis, geometry) in
the following sense: the relations that hold between the objects of one of these math-
ematical disciplines are represented in a completely corresponding way by relations
that obtain in a sub-domain of Zermelo's set theory.[14]

In spite of this perspective, Hilbert reconsidered at the end of the 1920-lecture
his earlier attempt (published as (1905a)) to establish by mathematical proof
that no contradiction can be proved in formalized elementary number theory.
That had raised already then the issue, how proofs can be characterized and
subjected to mathematical investigation. It was only after the study of *Prin-
cipia Mathematica* that Hilbert had a properly general and precise concept of
(formal) proof available.

3   Rigorous proofs

Proofs are essential for developing any mathematical subject, vide Euclid in the
Elements or Dedekind in *Was sind und was sollen die Zahlen?*. In the intro-
duction to his *Grundgesetze der Arithmetik*, Frege distinguished his systematic
development from Euclid's by pointing to the list of explicit inference principles
for obtaining gapless proofs. As to Dedekind's essay he remarked polemically
that no proofs can be found in that work. Dedekind and Hilbert explicated
the "science of (natural) number" and "arithmetic (of real numbers)" in sim-
ilar ways; their theories start from the defining conditions for simply infinite
systems, respectively complete ordered fields. Dedekind writes in (1888):

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[14](Hilbert *1920b, p. 23). Here is the German text: "Die Theorie, welche sich aus der En-
twicklung dieses Axiomensystems in seine Konsequenzen ergibt, schliesst alle mathematischen
Theorien (wie Zahlentheorie, Analysis, Geometrie) in sich in dem Sinne, dass die Beziehun-
gen, welche sich zwischen den Gegenständen einer dieser mathematischen Disziplinen finden,
vollkommen entsprechend dargestellt werden durch die Beziehungen, welche in einem Teilge-
biete der Zermeloschen Mengenlehre stattfinden."

The relations or laws which are derived exclusively from the conditions [for a simply infinite system] and are therefore always the same in all ordered simply infinite systems, ... form the next object of the *science of numbers* or *arithmetic*.[15]

The term "derive" is left informal; hence Frege's critique. Exactly at this point enters logic in the restricted modern sense as dealing with formal methods for correct, truth-preserving inference.

3.1   Natural deductions

Underlying Dedekind's and Hilbert's descriptions is an abstract concept of *logical consequence*. Hilbert stated in 1891 during a famous stop at a Berlin railway station that in a proper axiomatization of geometry "one must always be able to say 'tables, chairs, beer mugs' instead of 'points, straight lines, planes'." This remark has been taken as claiming that the basic terms must be meaningless, but it is more adequately understood if it is put side by side with a remark of Dedekind's in a letter to Lipschitz written fifteen years earlier: "All technical expressions [can be] replaced by arbitrary, newly invented (up to now meaningless) words; the edifice must not collapse, if it is correctly constructed, and I claim, for example, that my theory of real numbers withstands this test." Thus, logical arguments leading from principles to derived claims cannot be severed by a re-interpretation of the technical expressions or, to put it differently, there are no counterexamples to the arguments.

Dedekind's and Hilbert's presentations are detailed, reveal the logical form of arguments, and reflect features of the mathematical structures. In the very first sentence of the Preface to his (1888), Dedekind programmatically emphasizes that "in science nothing capable of proof should be accepted without proof" and claims that only common sense ("gesunder Menschenverstand") is needed to understand his essay. But he recognizes also that many readers will be discouraged, when asked to prove truths that seem obvious and certain by "the long sequence of simple inferences that corresponds to the nature of our step-by-step understanding" (Treppenverstand).[16] Dedekind believes that there are only a few such simple inferences, but he does not explicitly list them. Looking for an expressive formal language and powerful inferential tools, Hilbert moved slowly toward a presentation of proofs in logical calculi. He and his students started in 1913 to learn modern logic by studying *Principia Mathematica*. During the winter term 1917–18 he gave the first course in mathematical logic proper and sketched, toward the end of the term, how to develop analysis in ramified type theory with the axiom of reducibility.[17]

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[15](Dedekind 1888, sec. 73). In the letter to Keferstein, on p. 9, Dedekind reiterates this perspective and requires that every claim "must be derived completely abstractly from the logical definition of [the simply infinite system] N".

[16](Dedekind 1888, p. IV). Dedekind continues: "Ich erblicke dagegen gerade in der Möglichkeit, solche Wahrheiten auf andere, einfacherere zurückzuführen, mag die Reihe der Schlüsse noch so lang und scheinbar künstlich sein, einen überzeugenden Beweis dafür, daß ihr Besitz oder der Glaube an sie niemals unmittelbar durch innere Anschauung gegeben, sondern immer durch eine mehr oder weniger vollständige Wiederholung der einzelnen Schlüsse erworben ist."

[17]That is usually associated with the book (Hilbert and Ackermann 1928) that was published only in 1928; however, that book takes over the structure and much of the content from these earlier lecture notes. See my paper (1999) and the forthcoming third volume of

So there is finally (in Göttingen) a way of building up gapless proofs in Frege's sense. However, Hilbert aimed for a framework in which mathematics can be formalized in a natural and direct way. The calculus of *Principia Mathematica* did not lend itself to that task. In the winter term 1921–22 he presented a logical calculus that is especially interesting for sentential logic. He points to the parallelism with his axiomatization of geometry: groups of axioms are introduced there for each concept, and that is done here for each logical connective. Let me formulate the axioms for just conjunction and disjunction:

$$A \mathbin{\&} B \to A \qquad\qquad ((A \to C) \mathbin{\&} (B \to C)) \to ((A \vee B) \to C)$$

$$A \mathbin{\&} B \to B \qquad\qquad A \to (A \vee B)$$

$$A \to (B \to A \mathbin{\&} B) \qquad\qquad B \to (A \vee B)$$

The simplicity of this calculus and its directness for formalization inspired the work of Gentzen on natural reasoning. It should be pointed out that Bernays had proved the completeness of Russell's calculus in his *Habilitationsschrift* of 1918 and had investigated rule-based variants. The proof theoretic investigations of, essentially, primitive recursive arithmetic in the 1921–22 lectures also led to a tree-presentation of proofs, what Hilbert and Bernays called "the resolution of proofs into proof threads" (die Auflösung von Beweisen in Beweisfäden).[18] The full formulation of the calculus and the articulation of the methodological parallelism to *Grundlagen der Geometrie* are also found in (Hilbert and Bernays 1934, pp. 63–64).

3.2 Strategies

Gentzen formulated *natural deduction calculi* using Hilbert's axiomatic formulation as a starting point and called them calculi of *natural reasoning* (natürliches Schließen); he emphasized that making and discharging assumptions were their distinctive features. Here are the Elimination and Introduction rules for the connectives discussed above and as formulated in (Gentzen 1936); the configurations that are derived with their help are sequents of the form $\Gamma \supset \psi$ with Γ containing all the assumptions on which the proof of ψ depends:

$$\frac{\Gamma \supset A \mathbin{\&} B}{\Gamma \supset A} \qquad \frac{\Gamma \supset A \mathbin{\&} B}{\Gamma \supset B} \qquad\qquad \frac{\Gamma \supset A \vee B \quad \Gamma, A \supset C \quad \Gamma, B \supset C}{\Gamma \supset C}$$

$$\frac{\Gamma \supset A \quad \Gamma \supset B}{\Gamma \supset A \mathbin{\&} B} \qquad\qquad \frac{\Gamma \supset A}{\Gamma \supset A \vee B} \qquad \frac{\Gamma \supset B}{\Gamma \supset A \vee B}$$

Hilbert's *Lectures on the Foundations of Mathematics and Physics.* — In the final section of his (2008), Wiedijk lists "three main revolutions" in mathematics: the introduction of *proof* in classical Greece (culminating in Euclid's *Elements*), that of *rigor* in the 19th century, and that of *formal mathematics* in the late 20th and early 21st centuries. The latter revolution, if it is one, took place in the 1920s.

[18] On account of this background, I assume, Gentzen emphasized in his dissertation and his first consistency proof for elementary number theory the dual character of introduction and elimination rules, but considered making and discharging assumptions as the most important feature of his calculi.

Gentzen and later Prawitz established normalization theorems for proofs in nd calculi.[19] As the calculi are complete, one obtains proof theoretically refined completeness theorems: if ψ is a logical consequence of Γ, then there is a *normal* proof of ψ from Γ. I reformulated the nd calculi as *intercalation calculi*[20] for which these refined completeness theorems can be proved semantically without appealing to a syntactic normalization procedure; see (Sieg and Byrnes 1998) for classical first-order logic as well as (Sieg and Cittadini 2005) for some non-classical logics, in particular, for intuitionist first-order logic.

The refined completeness results and their semantic proofs provide foundations to the systematic search for normal proofs in nd calculi. This is methodologically analogous to the use of completeness results for cut-free sequent calculi and was exploited in the pioneering work of Hao Wang.[21] The subformula property of normal and cut-free derivations is fundamental for mechanical search. The ic calculi enforce normality by applying the E-rules only on the left to premises and the I-rules only on the right to the goal. In the first case one really tries to "extract" a goal formula by a *sequence* of E-rules from an assumption in which it is contained as a strictly positive subformula. This feature is distinctive and makes search efficient, but it is in a certain sense just a natural systematization and logical deepening of the familiar forward and backward argumentation. Suitable strategies have been implemented and guide a complete search procedure for first-order logic, called AProS.[22] In Appendix A, I discuss examples of purely logical arguments.[23] The AProS strategies can be extended by E- and I-rules for definitions, so that the meanings of defined notions as well as those of logical connectives can be used to guide search. In this way we have developed quite efficiently the part of elementary set theory concerning Boolean operations, power sets, Cartesian products, etc. In Appendix B, the reader finds two examples of set theoretic arguments.

You might think, that is interesting, but what relevance do these considerations have for finding proofs in more complex parts of mathematics? To answer that question and put it into a broader context, let me first note that the history of such computational perspectives goes back at least to Leibniz, and that it can be illuminated by Poincaré's surprising view of Hilbert's *Grundlagen der Geometrie*. In his review of Hilbert's book, he suggested giving the axioms to a reasoning machine, like Jevons's logical piano, and observing whether all of geometry would be obtained. He wrote that such radical formalization might

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[19]The first version of Gentzen's dissertation was recently discovered by Jan von Plato in the Bernays Nachlass of the ETH in Zürich. It contains a detailed proof of the normalization theorem for intuitionist predicate logic; see (von Plato 2008).

[20]I discovered only recently that Beth in his (1958) employs "intercalate" (on p. 87) when discussing the use of lemmata in the proofs of mathematical theorems.

[21]See the informative and retrospective discussion in his (1984) and, perhaps, also the programmatic (1970). — Cf. also my (2007).

[22]Nd calculi were considered as inappropriate for theorem proving because of the seemingly unlimited branching in a backward search afforded by modus ponens (conditional elimination). The global property of normality for nd proofs could not be directly exploited for a locally determined backward search; hence, the intercalation formulation of natural deduction. The implementation of AProS can be downloaded at `http://caae.phil.cmu.edu/projects/apros/`

[23]In (Sieg and Field 2005, pp. 334–5), the problem of proving that $\sqrt{2}$ is not rational is formulated as a logical problem, and AProS finds a proof directly; cf. the description of the difficulties of obtaining such a proof in (Wiedijk 2008).

seem "artificial and childish", were it not for the important question of "completeness":

Is the list of axioms complete, or have some of them escaped us, namely those we use unconsciously? ... One has to find out whether geometry is a logical consequence of the explicitly stated axioms or, in other words, whether the axioms, when given to the reasoning machine, will make it possible to obtain the sequence of all theorems as output [of the machine].[24]

With respect to a sophisticated logical framework and under the assumption of the finite axiomatizability of mathematics, Poincaré's problem morphed into what Hilbert and others viewed in the 1920s as the *most important problem* of mathematical logic: the decision problem (*Entscheidungsproblem*) for predicate logic. Its special character was vividly described in a talk Hilbert's student Behmann gave in 1921:

For the nature of the problem it is of fundamental significance that as auxiliary means ... only the completely mechanical reckoning according to a given prescription [Vorschrift] is admitted, i.e., without any thinking in the proper sense of the word. If one wanted to, one could speak of mechanical or machine-like thinking. (Perhaps it can later even be carried out by a machine.)

Johann von Neumann argued against the positive solvability of the decision problem, in spite of the fact that — as he formulated matters in 1924 — "... we have no idea how to prove the undecidability". It was only twelve years later that Turing provided the idea, i.e., introduced the appropriate concept, for proving the unsolvability of the *Entscheidungsproblem.*

The issue for Turing was, What are the procedures a human being can carry out when mechanically operating as a computer?[25] In his classical paper *On computable numbers with an application to the Entscheidungsproblem*, Turing isolated the basic steps underlying a computer's procedures as the operations of a Turing machine. He then proved: There is no procedure that can be executed by a Turing machine and solves the decision problem. Using the concepts of general recursive and λ-definable functions, Church had also established the undecidability of predicate logic. The core of Church's argument was presented in Supplement II of *Grundlagen der Mathematik, vol. II.* However, it was not only expanded by later considerations due to Church and Kleene, but also deepened by local axiomatic considerations for the concept of a *reckonable function.*[26]

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[24](Poincaré 1902b, pp. 252–253).

[25]For Turing a "computer" is a human being carrying out a "calculation" and using only minimal cognitive capacities. The limitations of the human sensory apparatus motivate finiteness and locality conditions; Turing's supporting argument is not mathematically precise, and I don't think there is any hope of turning the analysis into a mathematical theorem. What one can do, however, is to exploit it as a starting point for formulating a general concept and establishing a representation theorem; cf. my paper (2008a).

[26]I distinguish local from global axiomatics. As an example of the former I discuss in part 4.1 an abstract proof of Gödel's incompleteness theorems. Other examples can be found in Hilbert's 1917-talk in Zürich, but also in contemporary discussions, e.g., Booker's report on L-functions in the Notices of the AMS, p. 1088. Booker remarks that many objects go by the name of L-function and that it is difficult to pin down exactly which ones are. He attributes then to A. Selberg an "axiomatic approach" consisting in "writing down the common properties of the known examples" — as axioms.

Hilbert and Bernays introduced reckonable functions informally as those number theoretic functions whose values can be determined in a "deductive formalism". They proved that, if the deductive formalism satisfies their *recursiveness conditions*, then the class of reckonable functions is co-extensional with that of the general recursive ones. (The crucial condition requires that the proof relation of the deductive formalism is primitive recursive.) Their concept is one way of capturing the "completely mechanical reckoning according to a given prescription" mentioned in the quotation from Behmann. Indeed, it generalizes Church's informal notion of calculable functions whose values can be determined in a logic and imposes the recursiveness condition in order to obtain a mathematically rigorous formulation. For us the questions are of course: Can a machine carry out this mechanical thinking? and, if a universal Turing machine in principle can, What is needed to *copy*, as Turing put it in 1948, aspects of mathematical thinking in such a machine? — Copying requires an original, i.e., that we have uncovered suitable aspects of the mathematical mind when trying to extend automated proof search from logic to mathematics.

4   Local axiomatics

At the end of his report on *Intelligent Machinery* from 1948, Turing suggested that machines might search for proofs of mathematical theorems in suitable formal systems. It was clear to Turing that one cannot just specify axioms and logical rules, state a theorem, and expect a machine to demonstrate the theorem. For a machine to exhibit the necessary intelligence it must "acquire both discipline and initiative". Discipline would be acquired by becoming (practically) a universal machine; Turing argued that "discipline is certainly not enough in itself to produce intelligence" and continued:

That which is required in addition we call initiative. This statement will have to serve as a definition. Our task is to discover the nature of this residue as it occurs in man, and try and copy it in machines. (p. 21)

The dynamic character of strategies constitutes but a partial and limited copy of human initiative. Nevertheless, local axiomatics that allows the expression of leading ideas together with a hierarchical organization that reflects the conceptual structure of a field can carry us a long way. Hilbert expressed his views in 1919 as follows, arguing against the logicists' view that mathematics consists of tautologies grounded in definitions:

If this view were correct, mathematics would be nothing but an accumulation of logical inferences piled on top of each other. There would be a random concatenation of inferences with logical reasoning as its sole driving force. But in fact there is no question of such arbitrariness; rather we see that the formation of concepts in mathematics is constantly guided by intuition and experience, so that mathematics on the whole forms a non-arbitrary, closed structure.[27]

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[27](Hilbert *1919, p. 5). Here is the German text: "Wäre die dargelegte Ansicht zutreffend, so müsste die Mathematik nichts anderes als eine Anhäufung von übereinander getürmten logischen Schlüssen sein. Es müsste ein wahlloses Aneinanderreihen von Folgerungen stattfinden, bei welchem das logische Schliessen allein die treibende Kraft wäre. Von einer solchen Willkür ist aber tatsächlich keine Rede; vielmehr zeigt sich, dass die Begriffsbildungen in der Mathematik beständig durch Anschauung und Erfahrung geleitet werden, sodass im grossen

Hilbert's grouping of the axioms for geometry in his (1899a) had the express purpose of organizing proofs and the subject in a conceptual way: parts of his development are marvelous instances of *local axiomatics*, analyzing which notions and principles are needed for which theorems.

4.1   Modern

The idea of local axiomatics can be used for individual mathematical theorems and asks, How can we prove this particular theorem or this particular group of theorems? Hilbert and Bernays used the technique in their *Grundlagen der Mathematik II* also outside a foundational axiomatic context: first for proving Gödel's incompleteness theorems and then, as I indicated at the end of section 3.2, for showing that the functions reckonable in formal deductive systems coincide with the general recursive ones. One crucial task has to be taken on for *local* as well as for *global axiomatics*, namely, isolating what is at the heart of an argument or uncovering its *leading (mathematical) idea*. That was proposed by Saunders MacLane in his Göttingen dissertation (of late 1933) and summarized in his (1935). MacLane emphasized that proofs are not "mere collections of atomic processes, but are rather complex combinations with a highly rational structure". When reviewing in 1979 this early logical work, he ended with the remark, "There remains the real question of the actual structure of mathematical proofs and their strategy. It is a topic long given up by mathematical logicians, but one which still — properly handled — might give us some real insight."[28] That is exactly the topic I am trying to explore.

As an illustration of the general point concerning the "rational structure" of mathematical arguments, I consider briefly the proofs of Gödel's incompleteness theorems. These proofs make use of the connection between the mathematics that is used to present a formal theory and the mathematics that can be formally developed in the theory. Three steps are crucial for obtaining the proofs, steps that go beyond the purely logical strategies and are merged into the search algorithm:

1. *Local axioms*: representability of the core syntactic notions, the diagonal lemma, and the Hilbert & Bernays derivability conditions.

2. *Proof-specific definitions*: formulating instances of existential claims, for example, the Gödel sentence for the first incompleteness theorem.

3. *Leading idea*: moving between object- and meta-theory, expressed by appropriate Elimination and Introduction rules (for example, if a proof of A has been obtained in the object-theory, then one is allowed to introduce the claim 'A is provable' in the meta-theory).

AProS finds the proofs efficiently and directly, even those that did not enter into the analysis of the leading idea, for example, the proof of Löb's theorem. All of this is found in (Sieg and Field 2005).

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und ganzen die Mathematik ein willkürfreies, geschlossenes Gebilde darstellt."

[28]The first quotation is from MacLane's (1935, p. 130), the second from his (1979, p. 66). The processes by means of which MacLane tries to articulate the "rational structure" of proofs should be examined in greater detail.

It has been a long-standing tradition in mathematics to give and to analyze a variety of arguments for the same statement; the fundamental theorems of algebra and arithmetic are well-known examples. In this way we delimit conceptual contexts, provide contrasting *explanations* for the theorem at hand, and gain a deeper understanding by looking at it in different ways, e.g., from a topological or algebraic perspective.[29] An automated search requires obviously a sharp isolation of local axioms and leading ideas that underlie a proof. Such developments can be integrated into a global framework through a hierarchical organization, and that has been part and parcel of mathematical practice. Hilbert called it *Tieferlegung der Fundamente*!

These broad ideas are currently being explored in order to obtain an automated proof of the Cantor-Bernstein theorem from Zermelo's axioms for set theory.[30] The theorem claims that there is a bijection between two sets, in case there are injections from the first to the second and from the second to the first. The theorem is a crucial part of the investigations concerning the size of sets and guarantees the anti-symmetry of the partial ordering of sets by the "smaller-or-equal-size" relation.[31] We have begun to develop set theory from Zermelo's axioms and use three layers for the conceptual organization of the full proof:

A. Construction of sets, for example, empty set, power set, union, and pairs.

B. Introduction of functions as set theoretic objects.

C. The abstract proof.

The abstract proof is divided in the same schematic way as that of Gödel's theorems and is independent of the set theoretic definition of function. The *local axioms* are lemmata for injective, surjective, and bijective functions as well as a fixed-point theorem. The crucial *proof-specific definition* is that of the bijection claimed to exist in the theorem. Finally, the *leading idea* is simply to exploit the fixed-point property and verify that the defined function is indeed a bijection. — It is noteworthy that the differences between the standard proofs amount to different ways of obtaining the smallest fixed-point of an inductive definition.

4.2 Classical

Shaping a field and its proofs by concepts is classical; so is the deepening of its foundations. That can be beautifully illustrated by the developments in
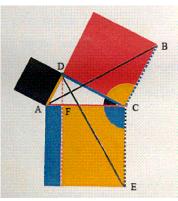
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[29]In the Introduction to the second edition of (Dirichlet 1863) Dedekind emphasized this aspect for the development of a whole branch of mathematics. In the tenth supplement to this edition of Dirichlet's lectures, he presented his general theory of ideals in order, as he put it, "to cast, from a higher standpoint, a new light on the main subject of the whole book". In German, "Endlich habe ich in dieses Supplement eine allgemeine Theorie der Ideale aufgenommen, um auf den Hauptgegenstand des ganzen Buches von einem höheren Standpunkte aus ein neues Licht zu werfen." He continues, "hierbei habe ich mich freilich auf die Darstellung der Grundlagen beschränken müssen, doch hoffe ich, daß das Streben nach charakteristischen Grundbegriffen, welches in anderen Teilen der Mathematik mit so schönem Erfolg gekrönt ist, mir nicht ganz mißglückt sein möge." (Dedekind 1932, pp. 396–7).

[30]My collaborators on this particular part of the AProS Project have been Ian Kash, Tyler Gibson, Michael Warren, and Alex Smith.

[31]On p. 209 of Cantor's (1932) *Gesammelte Abhandlungen*, Zermelo calls this theorem "one of the most important theorems of all of set theory".

the first two books of Euclid's *Elements* (and the related investigations at the beginning of Book XII). Proposition 47 of Book I, the Pythagorean theorem, is at the center of those developments. The broad mathematical context is given by the *quadrature problem*, i.e., determining the "size" or, in modern terms, the area of geometric figures in terms of squares. The problem is discussed in Book II for polygons. Polygons can be partitioned into triangles that can be transformed individually (by ruler and compass constructions) first into rectangles "of equal area" and then into equal squares.[32] The question is, how can we join these squares to obtain one single square that is equal to the polygon we started out with? It is precisely here that the Pythagorean theorem comes in and provides the most direct way of determining the larger square. Byrne's colorful diagram, displayed below, captures the construction and the abstract proof of the theorem. If one views the determination of the larger square as a geometric computation, then the proof straightforwardly verifies its correctness.[33]

For the proof, Euclid has us first construct the squares on the triangle's sides and then make the observation that the extensions of the sides of the smaller squares by the contiguous sides of the original triangle constitute lines. In the next step a crucial auxiliary line is drawn, namely, the line that is perpendicular to the hypotenuse and that passes through the vertex opposite the hypotenuse. This auxiliary line partitions the big square into the blue and yellow rectangles. Two claims are now considered: the blue rectangle is equal to the black square, and the yellow rectangle is equal to the red square. Euclid

uses three facts that are readily obtained from earlier propositions: (α) Triangles are equal when they have two equal sides and when the enclosed angles are equal (Proposition I.4); (β) Triangles are equal when they have the same base

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[32]Euclid simply calls the geometric figures "equal". This is central and has been pursued throughout the evolution of geometry. In Hilbert's (1899a), a whole chapter is devoted to "Die Lehre von den Flächeninhalten in der Ebene", Chapter IV, making the implicit Euclidean assumptions concerning "area" explicit. See also Hartshorne's book, section 22, *Area in Euclid's geometry*.

[33]Hilbert remarked in his (*1899b), "Wir werden im folgenden häufig Gebrauch von Figuren machen, wir werden uns aber niemals auf sie verlassen. Stets müssen wir dafür sorgen, daß die an einer Figur vorgenommenen Operationen auch rein logisch gültig bleiben. Es kann dies garnicht genug betont werden; im richtigen Gebrauch der Figuren liegt eine Hauptschwierigkeit unserer Untersuchungen." (p. 303) Here I am obviously not so much interested in the (correct) use of diagrams as analyzed by Manders and for which Avigad e.a. have provided an informative formal framework. Manders's analysis led to the assertion that only topological features of diagrams are relevant for and appealed to in Euclidean proofs; the conceptual setting sketched above with its focus on "area" gives a reason for that assertion. There is also important work from the early part of the 20th century by Hans Brandes and, in particular, Paul Mahlo in his 1908 dissertation. This work tries to classify the "Zerlegungsbeweise" of the Pythagorean theorem and should be investigated carefully; (Bernstein 1924) reflects on those dissertations.

and when their third vertex lies on the same parallel to that base (Proposition I.37); (γ) A diagonal divides a rectangle into two equal triangles (Proposition I.41).[34]

Here is the proof based on (α) through (γ) for the red square and the yellow rectangle. (The common notions are implicitly appealed to; the argument for the equality of the black square and the blue rectangle is analogous.) The triangles ABC and DCE satisfy the conditions of (α) and are thus equal; on account of (β) they are equal to DBC and FCE, respectively. Finally, (γ) ensures the equality of the red square and yellow rectangle. Comparing the structure of this argument to that of the abstract proofs for the incompleteness theorem and the Cantor-Bernstein theorem, we can make the following general observations: (α) through (γ) are used as *local axioms*; the auxiliary line drawn through the vertex opposite of the hypotenuse and perpendicular to it is the central *proof-specific definition*; finally, the *leading idea* is the partitioning of squares and establishing that corresponding parts are equal.

The character of the "deepening of the foundations" is amusingly depicted by anecdotes concerning Hobbes and Newton: Hobbes started with Proposition 47 and was convinced of its truth only after having read its proof and all the (proofs of the) propositions supporting it; Newton, in contrast, started at the beginning and could not understand, why such evident propositions were being established — until he came to the Pythagorean theorem. Less historically, there is also a deeper parallelism with the overall structure of the proof of the Cantor-Bernstein theorem from Zermelo's axioms. The construction of figures like triangles and squares corresponds to A (in the list A–C concerning the Cantor-Bernstein theorem); the congruence criteria for such figures correspond to B; the abstract proof of the geometric theorem, finally, has the same conceptual organization as the set theoretic proof referenced in C.

The abstract proof of the Pythagorean theorem and its deepening are shaped by the mathematical context, here the quadrature problem. I want to end this discussion with two related observations. Recall that the Pythagorean theorem is used in Hippocrates's proof for the quadrature of the lune.[35] This is just one of its uses for solving quadrature problems, but it seems to be very special, as only the case for isosceles triangles is exploited. The crucial auxiliary line divides in half the square over the hypotenuse, and we have a perfectly symmetric configuration.[36] Here is the first observation, namely, the claim concerning the equality of the rectangles (into which the square over the hypoteneuse is divided) and squares (over the legs) is "necessary", and the proof idea is relatively straightforward. That leads me to the second observation that is speculative and formulated as a question: Isn't it plausible that the Euclidean proof is obtained by generalizing this special one?

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[34]This is not *exactly* Euclid's proof. Euclid does not appeal to I.37, but just to I.41, which is really a combination of (β) and (γ) and applies directly to the diagram; I.37 is used in the proof of I.41. — The colorful diagram is from Byrne's edition of the first six books of the Euclidean *Elements*, London, 1847. [The labeling of points was added by me; WS.]

[35]See the very informative discussion in (Dunham 1990).

[36]In (Aumann 2009, pp. 64–65) knowledge of this geometric fact is attributed to the Babylonians, and it is the one Socrates extracts from the slave boy in Plato's *Meno*.

Karzel and Kroll, in their *Geschichte der Geometrie seit Hilbert* under the heading "Order and Topology", link the classical Greek considerations back (or rather forward) to modern developments:

In Euclidean geometry triangles and also rectangles take on the role of elementary figures out of which more complex figures are thought to be composed. To these elementary figures one can assign in Euclidean geometry an area in a natural way. If one assumes in addition the axiom of continuity, then one arrives at the concept of an integral when striving to assign an area also to more complex figures.[37]

So we have returned to continuity and to Dedekind.

5 Cognitive aspects

In his (1888) Dedekind refers to his *Habilitationsrede* where he claimed that the need to introduce appropriate notions arises from the fact that our intellectual powers are imperfect. Their limitation leads us to frame the object of a science in different forms and introducing a concept means, in a certain sense, formulating a hypothesis on the inner nature of the science. How well the concept captures this inner nature is determined by its usefulness for the development of the science, and in mathematics that is mainly its usefulness for constructing proofs. Dedekind put the theories from his foundational essays to this test by showing that they allow the direct, stepwise development of analysis and number theory. Thus, Dedekind viewed general concepts and general forms of arguments as tools to overcome, at least partially, the imperfection of our intellectual powers. He remarked:

Essentially, there would be no more science for a man gifted with an unbounded understanding — a man for whom the final conclusions, which we obtain through a long chain of inferences, would be immediately evident truths; and this would be so even if he stood in exactly the same relation to the objects of science as we do. (Ewald 1996, pp. 755–6)

The theme of bounded human understanding is sounded also in a remark from (Bernays 1954): "Though for differently built beings there might be a different kind of evidence, it is nevertheless our concern to find out what evidence is for us."[38] Bernays put forth the challenge of finding out what is evidence *for us*, not for some *differently built being.* Turing in his (1936) appealed crucially to *human* cognitive limitations to arrive at his notion of computability. Ten years later Gödel took the success of having given "an absolute definition of an interesting epistemological notion", i.e., of effective calculability, as encouragement to strive for "the same thing" with respect to demonstrability and mathematical definability. That was attempted in his (1946). Reflecting on a possible objection to his concept of ordinal definability, namely, that uncountably many sets are ordinal definable, Gödel considers as plausible the view

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[37](Karzel and Kroll 1988, p. 121). The German text is: "In der euklidischen Geometrie spielen neben den Dreiecksflächen noch die Rechtecksflächen ... die Rolle von Elementarflächen, aus denen man sich kompliziertere Flächen zusammengesetzt denkt. Diesen Elementarflächen kann man in der euklidischen Geometrie in natürlicher Weise einen Flächeninhalt zuweisen. Setzt man nunmehr noch das Stetigkeitsaxiom voraus, so gelangt man beim Bemühen, auch komplizierteren Flächen einen Inhalt zuzuweisen, zum Integralbegriff."

[38](Bernays 1954, p. 18). The German text is: "Obwohl es für anders gebildete Wesen eine andere Evidenz geben könnte, so ist jedoch unser Anliegen festzustellen, was Evidenz für uns ist."

"that all things conceivable by us are denumerable". Indeed, he thinks that a concept of definability "satisfying the postulate of denumerability" is possible, but "that it would involve some extramathematical element concerning the psychology of the being who deals with mathematics".

Reflections on cognitive limitations motivated also the finitist program's goal of an absolute epistemological reduction. Bernays provides in his (1922) a view of the program in *statu nascendi* and connects it to the existential axiomatics discussed above in Part 2. When giving a rigorous foundation for arithmetic or analysis one proceeds axiomatically, according to Bernays, and assumes the existence of a system of objects satisfying the structural conditions expressed by the axioms. In the assumption of such a system "lies something transcendental for mathematics, and the question arises, which principled position is to be taken [towards that assumption]". An intuitive grasp of the completed sequence of natural numbers, for example, or even of the manifold of real numbers is not excluded outright. However, taking into account tendencies in the exact sciences, one might try "to give a foundation to these transcendental assumptions in such a way that only primitive intuitive knowledge is used". That is to be done by giving finitist consistency proofs for systems in which significant parts of mathematics can be formalized. The second incompleteness theorem implies, of course, that such an absolute epistemological reduction cannot be achieved. What then is evidence for principles that allow us to step beyond the finitist framework? — Bernays emphasized in his later writings that evidence is acquired by *intellectual experience* and through *experimentation* in an almost Dedekindian spirit. In his (1946) he wrote:

In this way we recognize the necessity of something like intelligence or reason that should not be regarded as a container of [items of] a priori knowledge, but as a mental activity that consists in reacting to given situations with the formation of experimentally applied categories.[39]

This intellectual experimentation in part supports the introduction of concepts to define abstract structures or to characterize accessible domains (obtained by general inductive definitions), and it is in part supported by using these concepts in proofs of central theorems.[40]

I intended to turn attention to those aspects of the mathematical mind that are central, if we want to grasp the subtle connection between reasoning and

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[39](Bernays 1946, p. 91). The German text is: "Wir erkennen so die Notwendigkeit von etwas wie Intelligenz oder Vernunft, die man nicht anzusehen hat als Behältnis von Erkenntnissen a priori, sondern als eine geistige Tätigkeit, die darin besteht, auf gegebene Situationen mit der Bildung von versuchsweise angesetzten Kategorien zu reagieren." — Unfortunately, "applied" is not capturing "angesetzten". The latter verb is related to "Ansatz". That noun has no adequate English rendering either, but is used (as in "Hilbertscher Ansatz") to express a particular approach to solving a problem that does however not guarantee a solution.

[40]Andrea Cantini expresses in his recent (2008) a similar perspective, emphasizing also the significance of "geistiges Experimentieren" in Bernays's reflections on mathematics; see pp. 34–37. In the very same volume in which Cantini's article is published, Carlo Celluci describes a concept of *analytic* proof that incorporates many features of the experimentation both Cantini and I consider as important. However, Celluci sharply contrasts that concept with that of an *axiomatic* proof. These two notions, it seems to me, stand in opposition only if one attaches to the latter concept a dogmatic foundationalist intention. — In (Sieg 2010b) I have compared Gödel's and Turing's approach to such intellectual experimentation.

understanding in mathematics, as well as the role of leading ideas in guiding proofs and of general concepts in providing explanations. Implicitly, I have been arguing for an *expansion of proof theory*: Let us take steps toward a theory that articulates principles for organizing proofs conceptually and for finding them dynamically. A good start is a thorough reconstruction of parts of the rich body of mathematical knowledge that *is* systematic, but is also structured for intelligibility and discovery, when viewed from the right perspective. Such an expanded proof theory should be called *structural* for two reasons. On the one hand one exploits the intricate internal structure of (normal) proofs, and on the other hand one appeals to the notions and principles characterizing mathematical structures. (Cf. also the very tentative parallel remarks in Appendix C.)

When focusing on formal methods and carrying out computations in support of proof search experiments, we have to isolate truly creative elements in proofs and thus come closer to an understanding of the technique of our mathematical thinking, be it mechanical or non-mechanical. Hilbert continued his remarks in (1927) about the formula game as follows:

Thinking, it so happens, parallels speaking and writing: we form statements and place them one behind another. If any totality of observations and phenomena deserves to be made the object of a serious and thorough investigation, it is this one — since, after all, it is part of the task of science to liberate us from arbitrariness, sentiment, and habit ... (p. 475)

I could not agree more (with the second sentence in this quotation) and share Hilbert's eternal optimism, "Wir müssen wissen! Wir werden wissen!"

Appendices

AProS' distinctive feature is its *goal-directed* search for *normal* proofs. It exploits an essential feature of normal proofs, i.e., the division of every branch in their representing tree into an E- and an I-part; see (Prawitz 1965, p. 41). This global property of nd proofs, far from being an obstacle to backward search, makes proof search both strategic and efficient. — Siekmann and Wrightson collected in their two volume *Automated Reasoning* classical papers that contain marvelous discussions of the broad methodology underlying different approaches in the emerging field from the late 1950s to the early 1970s. The papers by Beth, Kanger, Prawitz, Wang and the "Russian School" are of particular interest from my perspective, as we find in them serious attempts of searching for humanly intelligible proofs and of getting the logical framework right before building heuristics into the search. That was perhaps most clearly formulated by Kanger in his (1963, p. 364): "The introduction of heuristics may yield considerable simplifications of a given proof method, but I have the impression that it would be wise to postpone the heuristics until we have a satisfactory method to start with." The work with AProS and automated proof search support that view.

A. Purely logical arguments.

In the supplement to (Shanin, et al. 1965), one finds five propositional problems and their proofs; AProS solves them with just the basic rules whereas in this

paper quite complex derived rules are used. I discuss one example in order to illustrate, how dramatically the search is impacted by "slight" reformulations of the problem to be solved or, what amounts to the same thing, by introducing specific heuristics. The problem in (Shanin, et al. 1965) is to derive

$$(\neg(K \to A) \vee (K \to B))$$

from the premises

$$(H \vee \neg(A \ \& \ K)) \text{ and } (H \to (\neg A \vee B)).$$

The pure AProS search procedure uses 277 search steps to find a proof of length 77. If one uses in a first step a derived rule to replace positive occurrences of $(\neg X \vee \Delta)$ or $(X \vee \neg\Delta)$ by $(X \to \Delta)$, respectively, $(\Delta \to X)$ then AProS uses 273 search steps for a proof of length 87 (having made the replacement in the first premise), 149 search steps for a proof of length 80 (having made the replacement also in the second premise), and finally 9 search steps to find a derivation of length 12 (having made the replacement also in the conclusion).

The Shanin-procedure introduces also instances of the law of excluded middle. In the above problem it does so for the left disjunct of the goal, i.e., it uses the instance $(\neg(K \to A) \vee (K \to A))$. If one adds that instance as an additional premise, then AProS takes 108 search steps to obtain a proof of 49 lines. If the conclusion $(\neg(K \to A) \vee (K \to B))$ is replaced by $((K \to A) \to (K \to B))$ then the instance of the law of excluded middle is not used when AProS obtains a proof of length 29 in 23 search steps. If only the goal is reformulated as a conditional, then the same proof is obtained with just 18 search steps.

The replacement step in the last proof amounts to using one of the available rules from (Shanin, et al. 1965) heuristically: if the goal is of the form $(\neg X \vee \Delta)$ or $(X \vee \neg\Delta)$ then prove instead $(X \to \Delta)$, respectively, $(\Delta \to X)$. Such a reformulation of a problem, or equivalently the strategic use of a derived rule, can thus have a dramatic consequence on the search and the resulting derivation. Let me discuss two additional examples and a motivated extension of this heuristic step:

(1) Prove from the premise $P \vee Q$ the disjunction

$$(P \ \& \ Q) \vee (P \ \& \ \neg Q) \vee (\neg P \ \& \ Q).$$

With its basic algorithm AProS uses 202 search steps to find a proof of length 58; however, if the goal is reformulated as the conditional

$$\neg(P \ \& \ Q) \to ((P \ \& \ \neg Q) \vee (\neg P \ \& \ Q))$$

then 28 steps lead to a proof of length 18.
(2) Prove $((P \vee Q) \to (P \vee R)) \to (P \vee (Q \to R))$.
103 steps in the basic search lead to a proof of length 47. If one considers instead

$$((P \vee Q) \to (P \vee R)) \to (\neg P \to (Q \to R))$$

AProS finds a proof of length 14 with 9 search steps.
These quasi-empirical observations can be used to articulate a heuristic for the purely logical search: if one encounters a disjunction $(X \vee \Delta)$ as the goal, prove instead the conditional $(\neg X \to \Delta)$ (and eliminate in the antecedent a double negation, in case X happens to be a negation).

B. Some elementary set theoretic arguments.

As I mentioned in sections 3.2 and 4.1, we have been extending the automated search procedure to elementary set theory. Though our goal is different from that of interactive theorem proving, there is a great deal of overlap: the hierarchical organization of the search can be viewed as reflecting and sharpening the interaction of a user with a proof assistant. After all, we start out by analyzing the structure of proofs, formalizing them, and then automating the proof search, i.e., completely eliminating interaction. The case of using computers as proof assistants is made in great detail in Harrison's paper (2008). For the case of automated proof search it is important, if not absolutely essential, that the logical calculus of choice is natural deduction.[41]

Natural deduction has been used for proof search in set theory; an informative description is found, for example, in (Bledsoe 1983). Pastre's (1976) dissertation, deeply influenced by Bledsoe's work, is mentioned in Bledsoe's paper. She has continued that early work, and her most recent paper (2007) addresses a variety of elementary set theoretic problems. Similar work, but in the context of the *Theorema* project, was done in (Windsteiger 2001) and (Windsteiger 2003). However, the term natural deduction is used here only in a very loose way: there is no search space that underlies the logical part and guarantees completeness of the search procedure. Rather, the search is guided in both logic and set theory by "natural heuristics" for the use of reduction rules that are not connected to a systematic logical search and, in Pastre's case, do not even allow for any backtracking.

Let me consider a couple of examples of AProS proofs to show how the logical search is extended in a most natural way by exploiting the meaning of defined concepts by appropriate I- and E-rules. That has, in particular, the "side-effect" of articulating in a mathematically sensible way, at which point in the search definitions should be expanded. In each case, the reader should view the proof strategically, i.e., closing the gap between premises and conclusion by use of (inverted) I-rules and motivated E-rules.

Example 1: $a \in b$ proves $a \subseteq \bigcup(b)$

1.	$a \in b$	Premise
2.	$\quad u \in a$	Assumption
3.	$\quad (a \in b \ \& \ u \in a)$	&I 1, 2
4.	$\quad (\exists z)(z \in b \ \& \ u \in z)$	$\exists$I 3
5.	$\quad u \in \bigcup(b)$	Def.I (Union) 4
6.	$(u \in a \rightarrow u \in \bigcup(b))$	$\rightarrow$I 5
7.	$(\forall x)(x \in a \rightarrow x \in \bigcup(b))$	$\forall$I 6
8.	$a \subseteq \bigcup(b)$	Def.I (Subset) 7

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[41]There have been attempts of using proofs by resolution or other "machine-oriented" procedures as starting points for obtaining natural deduction proofs; Peter Andrews and Frank Pfenning, but also more recently Xiaorong Huang did interesting work in that direction.

Example 2.1: $a \subseteq b$ proves $\wp(a) \subseteq \wp(b)$

1.	$a \subseteq b$	Premise
2.	$u \in \wp(a)$	Assumption
3.	$v \in u$	Assumption
4.	$(\forall x)(x \in a \to x \in b)$	Def.E (Subset) 1
5.	$(v \in a \to v \in b)$	$\forall$E 4
6.	$u \subseteq a$	Def.E (Power Set) 2
7.	$(\forall x)(x \in u \to x \in a)$	Def.E (Subset) 6
8.	$(v \in u \to v \in a)$	$\forall$E 7
9.	$v \in a$	$\to$E 8, 3
10.	$v \in b$	$\to$E 5, 9
11.	$(v \in u \to v \in b)$	$\to$I 10
12.	$(\forall x)(x \in u \to x \in b)$	$\forall$I 11
13.	$u \subseteq b$	Def.I (Subset) 12
14.	$u \in \wp(b)$	Def.I (Power Set) 13
15.	$(u \in \wp(a) \to u \in \wp(b))$	$\to$I 14
16.	$(\forall x)(x \in \wp(a) \to x \in \wp(b))$	$\forall$I 15
17.	$\wp(a) \subseteq \wp(b)$	Def.I (Subset) 16

Example 2.2: This is example 2.1 with the additional premise (lemma):

$$(\forall x)[(x \subseteq a \ \& \ a \subseteq b) \to x \subseteq b]$$

1.	$(\forall x)[(x \subseteq a \ \& \ a \subseteq b) \to x \subseteq b]$	Premise
2.	$a \subseteq b$	Premise
3.	$u \in \wp(a)$	Assumption
4.	$(u \subseteq a \ \& \ a \subseteq b) \to u \subseteq b$	$\forall$E 1
5.	$u \subseteq a$	Def.E (Power Set) 3
6.	$(u \subseteq a \ \& \ a \subseteq b)$	&I 5, 2
7.	$u \subseteq b$	$\to$E 4, 6
8.	$u \in \wp(b)$	Def.I (Power Set) 7
9.	$(u \in \wp(a) \to u \in \wp(b))$	$\to$I 8
10.	$(\forall x)(x \in \wp(a) \to x \in \wp(b))$	$\forall$I 9
11.	$\wp(a) \subseteq \wp(b)$	Def.I (Subset) 10

C. Confluence?

The AProS project intends also to throw some empirical light on the cognitive situation. With a number of collaborators I have been developing a web-based introduction to logic, called *Logic & Proofs*; it focuses on the strategically guided construction of proofs and includes dynamic tutoring via the search algorithm AProS. The course is an expansive Learning Laboratory, as students construct arguments in a virtual Proof Lab in which their every move is recorded. It allows the investigation of questions like:

- How do students go about constructing arguments?

- How do particular pedagogical interventions affect their learning?

- How efficient do students get in finding proofs with little backtracking?

- Does the skill of strategically looking for proofs transfer to informal considerations?

The last question hints at a broader and long-term issue I am particularly interested in, namely, to find out whether strategic-logical skills improve the ability of students to understand complex mathematics.

The practical educational aspects are deeply connected to a theoretical issue in cognitive science, namely, the stark opposition of "mental models" (Johnson-Laird) and "mental proofs" (Rips). I do not see an unbridgeable gulf, but consider the two views as complementary. Proofs as diagrams give rise to mental models, and the dynamic features of proof construction I emphasized are promoted by and reflect a broader structural, mathematical context; all of this is helping us to bridge the gap between premises and conclusion. The crucial question for me is: Can we make advances in isolating basic operations of the mind involved in constructing mathematical proofs or, in other words, can we develop a cognitive psychology of proofs that reflect logical and mathematical understanding?

There is deeply relevant work on analogical reasoning, e.g., Dedre Gentner's. In the (2008) manuscript with J. Colhoun they write, "Analogical processes are at the core of relational thinking, a crucial ability that, we suggest, is key to human cognitive prowess and separates us from other intelligent creatures. Our capacity for analogy ensures that every new encounter offers not only its own kernel of knowledge, but a potentially vast set of insights resulting from parallels past and future." Performance in particular tasks is enhanced when analogies, viewed as relational similarities, are strengthened by explicit comparisons and appropriate encodings. It seems that abstraction is here a crucial mental operation and builds on such comparisons. The underlying theoretical model of these investigations (structure mapping) is steeped in the language of the mathematics that evolved in the 19th century, in particular, through Dedekind's work. It was Dedekind who introduced *mappings* between arbitrary systems; he asserted in the strongest terms that without this *capacity of the mind* (to let a thing of one system correspond to a thing of another system) no thinking is possible at all. Modern abstract, structural mathematics, one can argue convincingly, makes analogies between different "structures" precise via appropriate axiomatic formulations. — All of this, so the rich psychological experimental work demonstrates, is important for learning. In the context of more sophisticated mathematics, *Kaminski e.a.* hypothesized (and confirmed) for example recently "that learning a single generic instantiation [i.e., a more abstract example of a structure or concept; WS] ... may result in better knowledge transfer than learning multiple concrete, contextualized instantiations." (p. 454)

There is a most plausible confluence of mathematical and psychological reflection that would get us closer to a better characterization of the "capacity of the human mind" that was discovered in Greek and rediscovered in 19th century mathematics; according to Stein, as quoted already at the beginning of Part 2, "what has been learned, when properly understood, constitutes one of the greatest advances in philosophy ..."

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The following abbreviations are being used: DMV for Deutsche Mathematiker Vereinigung; BSL for Bulletin of Symbolic Logic; AMS for American Mathematical Society.

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